CHAPTER

10 PROBABILITY

SKILLS TO KNOW

- Basic probability
- Finding the probability something will not happen
- Independent events & dependent events
- And situations/Or situations
- Probability & permutations
- Finding expected values
- Probability notation, union & intersection

NOTE: This chapter builds heavily on **Chapter 9** in this book, **Counting & Arrangements**. Be sure you have a handle on Chapter 9 before attempting this chapter.

BASIC PROBABILITY

The **probability** of an event occurring is the likelihood that something will happen.

Probability is expressed as a decimal or fraction between 0 and 1, inclusive. If the probability of an event is 1, it will happen with 100% certainty. The closer the probability of an event is to 1, the more likely it will occur. The closer the probability of an event is to zero, the less likely it is to occur.

When the selection of an outcome is **at random**, we can calculate probability by creating a fraction:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

For example, if we want to know the chances of choosing a blue marble from a box when there are 3 blue marbles and 10 marbles total, that would be $\frac{3}{10}$.

This can also be expressed as:

$$\frac{\textit{Number of "successful" outcomes possible}}{\textit{Number of "successful" outcomes possible} + \textit{Number of "failed" outcomes possible}}$$

For example, if there are 3 blue marbles and 7 other colored marbles, there are 3 ways to succeed and 7 ways to fail if I pick one marble out of the box and want a blue one. The probability is thus:

$$\frac{3}{3+7}$$

There are 52 cards in a deck of cards. There are 4 suits (spades, clubs, diamonds, and hearts), each with 13 cards, 3 of which are face cards. 2 suits are red, and 2 are black. If a card is drawn at random, what is the probability that it is a red face card?

To solve, we use the fraction formula for finding probability:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

We need to find two things:

1. **The numerator:** the number of red face cards (what we want).

   We have 3 face cards per suit. Two of the suits are red. That means we have $2(3)$ or 6 total red face cards.

2. **The denominator:** the total number of cards in the deck (total possible outcomes).

   Per the question, we have 52 cards in the deck. We now divide the value for #1 above by that of #2:

$$\frac{6}{52}=\frac{3}{26}$$

Answer: $\frac{3}{26}$.

PROBABILITY THAT SOMETHING *WON'T* HAPPEN

Finding the probability something *will* happen can also be solved by calculating the probability that something *won't* happen. In the world of probability, success and failure are mutually exclusive concepts: either something happens or it doesn't; two options exist. As a result, the probability that something will happen and that something won't happen always sum to 1. For example, if there is a $\frac{3}{5}$ chance you'll pick a red marble from a bucket, there is a $\frac{2}{5}$ chance you won't.

If you need to know the probability that something happens, but it's easier to find the probability of that something not happening, solve for the latter probability and subtract from one. Likewise, if you're asked to find the probability of something *not happening* and it's easier to solve for probability of something *happening*, solve for that and subtract from one.

> There are 100 slips of paper numbered 1 through 100 inclusive in a hat. If one slip is drawn at random, what is the probability the number drawn is not a perfect square?

I know $1,4,9,16,25,36,...,81,100$ are the list of perfect squares. I omitted the middle, because I know how each is formed: squaring a number 1 to 10 inclusive, as 1 is 1 squared, and 100 is 10 squared. In between, I'll have $2^2,3^2,4^2$, etc. Thus I know there will be 10 of these numbers in the list. As a result, the probability of getting a perfect square as my number on the slip is:

$$\frac{10}{100}=\frac{1}{10}$$

Now I subtract this value from 1 to find the answer: $1-\frac{1}{10}=\frac{9}{10}$.

Answer: $\frac{9}{10}$.

INDEPENDENT VS. DEPENDENT EVENTS

In the previous chapter, we defined **independent** and **dependent** events. To review:

Independent events don't affect the outcome of other events. For example, if I flip a penny and get heads, I'm not more or less likely to get heads again on a second flip. Coin flips, dice rolls, and situations with wording such as **"with replacement"** or **"items/digits can repeat"** are typically **independent events.**

Dependent events affect the probability of subsequent events. Drawing three letters from a bag of lettered tiles, choosing people for a team, or selecting songs to sing at a recital are all dependent events. You wouldn't sing the same song twice at your voice recital, so what you pick for the first song affects what you choose for the 2nd. If you had 4 songs to choose from, after you choose one song you'll only have 3 songs to choose from. Often, you'll see words like **"distinct," "unique,"** or **"without replacement"** when encountering problems that involve **dependent events**.

"AND" SITUATIONS

If the probability of two independent events are A and B, then the probability of both A "AND" B occurring (assuming each event is unique, or order matters) **is A times B.** In short, when you have an **"AND"** situation (Event A is true AND Event B is true) **you multiply.**

> Ned is throwing a coin to see if it's heads or tails. What is the probability that he will throw heads three times in a row?

This is an "AND" situation involving independent events. The chance of getting heads once is $\frac{1}{2}$. Since Ned will be doing this three times and order matters (to get three heads in a row we need each subsequent toss to be heads, the first, the second, and the third) we multiply $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$ to get $\frac{1}{8}$.

Answer: $\frac{1}{8}$.

A similar trend emerges with **dependent events**. We still **multiply probabilities together** when both **event A AND event B** are true to find the probability both are true (assuming **order matters**). However, we must take event A into account when we calculate the probability of B. In other words, we multiply **(Probability of Event A) (Probability of event B given event A happening first)**.

> Marlin is choosing three toys at random to take to the beach from her basket of 10 beach toys. If $\frac{1}{2}$ of the beach toys are plastic, what is the probability she chooses all plastic toys?

We start by remembering the fraction that defines probability:

$$\frac{\textit{Number of possible desired outcomes}}{\textit{Number of total possible outcomes}}$$

These are dependent events, so as I work I must take into account the previous choices. I want to find the probability that three particular events occur in a row: she chooses a plastic toy first, a plastic toy second, and a plastic toy third. For all of these events to be true, we have **an "AND" situation**. Each of these events **must occur** in order to get my desired outcome. Thus we can find the probability of each case and then must **multiply these individual probabilities together**.

Her probability of choosing a plastic toy the first time is $\frac{5}{10}$, but the second is $\frac{4}{9}$. When she chooses the 2nd toy, the 1st toy is no longer an option; there are only 4 plastic toys to choose from and 9 toys left in total to choose from. When she chooses the third toy, there is a $\frac{3}{8}$ chance she chooses a plastic one. At this point, she will only have 3 plastic toys left to choose from and 8 toys in total.

As you can see, we reduce the numerator and denominator accordingly as we go: $\frac{5}{10}\times\frac{4}{9}\times\frac{3}{8}=\frac{1}{12}$.

Why can we solve this problem using a "permutation" when it sounds like a combination?

Though order doesn't matter here in one sense (she is selecting a few items from a group), it DOES matter in terms of her selecting a toy that is plastic on each turn, i.e. each moment when she chooses a toy a certain event must occur: she picks a plastic toy each time. Thus we can still treat this as a permutation and not a combination. We could actually solve this problem using combinations as well. For that method, we'd rely again on the fraction that defines probability, and calculate our numerator as the number of ways to choose 3 plastic toys from 5 (${}_5C_3$ or "5 choose 3") and then divide that by the number of ways to choose 3 toys from 10 (${}_{10}C_3$ or "10 choose 3"). Using our calculator's built in combinations function we get: $\frac{{}_5C_3}{{}_{10}C_3}$ which equals $\frac{10}{120}$ or $\frac{1}{12}$.

"OR" SITUATIONS

Sometimes we can find a probability by adding together the probability of all the unique ways we could get what we want. These cases are essentially **"OR" scenarios**. Situation A is true or B is true or C is true, for example. When we have **"OR"** situations in probability, and our elements are unique (i.e. "mutually exclusive") we **ADD the probabilities together.** Add the probability of all the unique cases that produce your desired outcome to find the overall probability of that outcome.

Let's say we want to know the probability of flipping one head and one tail when two coins are flipped. For example, if the chance I get heads then tails on two coin flips is $\frac{1}{4}$ and the chance I get tails then heads on two coin flips is $\frac{1}{4}$, then the chance in two flips of a coin that result in one heads and one tails is $\frac{1}{4}+\frac{1}{4}$ or $\frac{1}{2}$.

WARNING: If you add probabilities together, you must be certain the outcomes included within each probability you've calculated **DO NOT OVERLAP**. Events must be distinct or **"mutually exclusive"** if you're going to add their probabilities. I can't be in 4th grade and in 5th grade, those are mutually exclusive events that do not overlap. But I could be in 4th grade and female. Those are NOT mutually exclusive events. So if I know 1/2 the students at a school are female and 1/5 are in 4th grade, I CANNOT simply add 1/2 plus 1/5 to find the probability of selecting a student at random who is female and/or in fourth grade. If I did, I would double count all the fourth grade girls.

> A multiple-choice quiz has four answer options for each of five questions. What is the probability of choosing answer choices at random and missing exactly one question?

Because getting a question right or wrong is independent of how I did on the last question (assuming I'm guessing at random), the events are independent. To solve this problem, I'll pretend order matters and break it into multiple cases in which order matters that produce the combination we want.

To get four right and one wrong could look like this:

$$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{1024}$$

This is the probability that I miss ONE question, and that the question I miss is the LAST question. But there are 5 different orders this could happen in, i.e. the "wrong question" (our $\frac{3}{4}$ in the string above) could also be 1st, 2nd, 3rd, or 4th:

Case 1: I get question 1 wrong:

$$\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{1024}$$

Case 2: I miss question 2:

$$\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{1024}$$

Case 3: I miss question 3:

$$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{1024}$$

Case 4: I miss question 4:

$$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{3}{1024}$$

For each of these cases the fractions are the same as the case I initially wrote out, but just in a different order. Each of the five cases has a probability of $\frac{3}{1024}$.

This is an "or" situation because any of the above options will work and none will overlap. I thus need the sum of all these possibilities. I could add $\frac{3}{1024}$ to itself five times, or simply multiply $\frac{3}{1024}$ by five to get the answer:

$$5\left(\frac{3}{1024}\right)=\frac{15}{1024}$$

Answer: $\frac{15}{1024}$.

USING PERMUTATIONS IN PROBABILITY PROBLEMS

More complex probability problems synthesize your knowledge of counting problems and of probability. To solve these problems, remember probability is always found by finding:

$$\frac{\textit{Total number of outcomes that fulfill the desired parameters}}{\textit{Total number of possible outcomes}}$$

Oftentimes, we can use permutations (or the fundamental counting principle or even combinations) to solve for each of these two values, and then in turn solve for the probability. The ACT® rarely requires you to know how to do these with combinations, so I'll focus on permutations. Still, the same principle would work if the problem you confront involves combinations.

In the bleachers of a football stadium, 2 boys and 3 girls are seated together in a random order. What is the probability that the 2 boys are seated next to each other?

First, we know we need to solve for two values:

1. Total number of ways to arrange 2 boys and 3 girls such that the 2 boys are always next to each other (numerator of our probability)
2. Total number of ways to arrange 5 kids (I could say 2 boys and 3 girls, but each person is actually unique, so this is really just how to arrange 5 kids; thinking this way makes the math easier).

Both of these elements can be solved using permutations and a bit of creativity.

Let's start with #1:

When I need to keep 2 items next to each other in a permutation, one way I can think of this is Case 1/Case 2. Let's name our boys Brian and Max, and our girls Leah, Wei Wei, and Ann.

Case 1: Brian is seated directly to the left of Max
Case 2: Brian is seated directly to the right of Max

Now I can solve for the number of permutations I have, pretending that Brian and Max are "glued" together and essentially are one person. I'll just calculate the number of ways to arrange for each case and add all the possibilities together.

Case 1: I can choose from the following four taken four at a time:

Brian/Max, Leah, Wei Wei, Ann

$$\underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } = 24$$

Case 2: I can choose from the following four taken four at a time:

Max/Brian, Leah, Wei Wei, Ann

$$\underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } = 24$$

I add these together and get 48 (I also could have seen that Case 1 & 2 will have the same number of options, and thus could have simply multiplied 24 by 2). In any case, I know my numerator: 48.

Now for Step #2:
How many ways can I arrange 5 kids, in order? 5 kids taken 5 at a time when order matters is simply:

$$\underline{\ 5\ } \times \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } = 120$$

120 is my denominator.

Now I simplify the fraction:

$$\frac{\textit{Total number of outcomes that fulfill the desired parameters}}{\textit{Total number of possible outcomes}}$$

$$= \frac{48}{120} = \frac{2}{5} \ or\ .4$$

EXPECTED VALUES

Finding the expected value is like finding a weighted average. Here's an example.

The probability distribution of the discrete random variable Y is shown in the table below. What is the expected value of Y?

y	Probability $P(Y=y)$
0	0.15
1	0.26
2	0.29
3	0.11
4	0.19

First of all, don't be thrown by the language **"discrete random variable."** That just means that Y is not continuous as a possible value. For example, the number of people in an elevator is a discrete number because you can't have half a person; every number is a whole number. If I listed out probabilities of the number of people in the office elevator on any given trip, my values would all be discrete. Discrete variables don't have to be integers, but the point is that you don't have to worry about a bunch of values in between what is on your chart.

MISTAKE ALERT! Whenever I have problems with a probability chart like this, I always double check that the given probabilities add to one. If not, the chart is not a complete depiction of what is going on and I must account for that. Occasionally these problems will only give you the "first few" or "select" values of this variable and not all of them.

Here, I see that all my probabilities sum to 1. Thus I know to find my expected value, I simply multiply the value of y times its probability of occurring. Then I add all these little values up:

$$0(0.15)+1(0.26)+2(0.29)+3(0.11)+4(0.19)$$
$$=0+0.26+0.58+0.33+0.76$$
$$=1.93$$

The answer should essentially be the weighted average of the values in your chart. If your answer doesn't seem about "average," go back and check your work. Here this makes sense. 1 and 2 occur most often.

Answer: 1.93.

PROBABILITY NOTATION, UNION & INTERSECTION

(**FYI, this is NOT frequently tested.** This section is for overachievers with lots of time only!).

Occasionally, the ACT® may use certain notation to denote probability. Typically, however, the ACT® will define this notation for you if you are expected to use it. That means you shouldn't worry too much about remembering everything below; just be familiar with it.

We say that the probability of Event A occurring is $P(A)$. What that means is that if I write "$P(A)$" that represents the fraction or decimal probability that something happens. Similarly the probability of Event B occurring would be $P(B)$, of Event C, $P(C)$ and so on.

> Let $P(A)$ represent the probability of event A occurring. If event R occurs when three coins are flipped and all are heads, calculate $P(not\ R)$.

For this problem, we can first calculate the odds of getting all heads by multiplying the independent probability of each event. Since the probability of flipping heads is $\frac{1}{2}$, we multiply:

$$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

Because we need "not R" we now subtract the probability of getting R from 1:

$$1-\frac{1}{8}=\frac{8}{8}-\frac{1}{8}=\frac{7}{8}$$

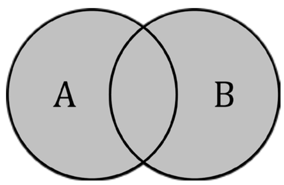
Answer: $\frac{7}{8}$.

"UNION"

You may occasionally see a U-like symbol in probability problems. When it is facing upwards, we call this "union." For example, the probability of A union B is denoted by:

$$P(A\cup B)$$

Union is the probability that **A happens, B happens, or both happen**. It is always greater than or equal to the probability of A alone or B alone. The picture below is one way to visualize A union B if there is some overlap.



Unlike the "mutually exclusive" situations discussed earlier, union situations often involve overlap (though they need not involve overlap). For example, if event A is having brown hair and event B is having blue eyes, some people have both traits, and that would be "overlap" if we counted those who had A **OR** B. **When events have some overlap we call these "inclusive events."**

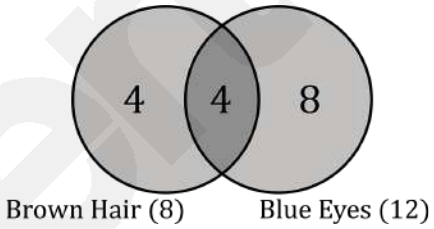
If Events A and B are inclusive, then the probability that A or B occurs is the sum of their probabilities minus the probability that both occur; i.e. to find the combined probability of inclusive events, we add the individual probabilities together and subtract the overlap:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

To help understand the formula, draw a **Venn diagram** to visualize, as in the problem below.

In in a class of 24 students, if 12 students have blue eyes, the probability of which is denoted by $P(A)$, 8 students have brown hair, the probability of which is denoted by $P(B)$, and 4 students have both, the probability of which is denoted by $P(A \text{ and } B)$, what is the probability that students have brown hair or blue eyes, denoted by $P(A \text{ or } B)$?

First, don't be thrown by all the notation. It's only there to confuse you. I know probability is the number of desired outcomes divided by the possible outcomes. I know I have 24 kids in the class, so that is my denominator. My numerator is the number of blue eyed and brown haired students *inclusive*, i.e. anyone who has either trait or both: those with brown hair and not blue eyes, those with both blue eyes and brown hair, and those with blue eyes and not brown hair. I can start by figuring out each of these cases using the Venn Diagram below. I subtract 4 (overlap) from 8 (number of students who have brown hair) to find the number who have brown hair but not blue eyes and subtract 4 from 12 to find blue-eyed kids without brown hair (8):



I can now add each segment in my Venn Diagram to find the number of people who have either brown hair, blue eyes, or both ($4+4+8=16$).

I can also find this by taking $8+12-4=16$, per the formula we discussed earlier.

Now I place 16 over 24: $\frac{16}{24}=\frac{4}{6}=\frac{2}{3}$.

Answer: $\frac{2}{3}$.

NOTE: For more on Venn Diagrams see the **Chapter 11: Word Problems** in Book 1.

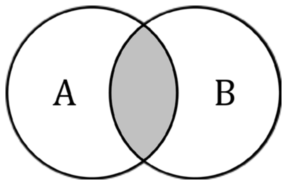
"INTERSECTION"

Another notation you might see is an upside down "U." We call this an intersection. The probability of A intersection B is denoted by:

$$P\left(A \cap B\right)$$

An intersection occurs when **A AND B** are both simultaneously true. It is always less than or equal to the probability of A or B alone.

The picture below depicts what A intersection B looks like; it is the "overlap" of values in both sets:



Let A and B be independent events. Denote $P(A)$ as the probability that event A will occur, $P(A \cup B)$ as the probability that event A or B or both will occur. Which of the following equations *must* be true? (Note: $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.)

A. $P(A)P(B) = P(A \cup B)$

B. $P(A) > P(A \cup B)$

C. $P(B) - P(A) = P(A \cap B) - 2P(A) + P(A)P(B)$

D. $P(A \cup B) > P(A) + P(B)$

E. $P(A)P(A \cup B) = P(A)^2 - P(A)^2 P(B) + P(A)P(B)$

Let's go through each choice:

A. $P(A)P(B) = P(A \cup B)$

When we multiply two independent probabilities together, we find the chances that *both* occur. Thus this is a calculation of the intersection, NOT the union. A is incorrect.

B. $P(A) > P(A \cup B)$

We know that union essentially adds any items in set B to set A. Thus the probability of A alone cannot be greater than a probability that has at least as many if not more options that make it true. B is also incorrect.

C. $P(B) - P(A) = P(A \cap B) - 2P(A) + P(A)P(B)$

At first, this might look like an algebraic manipulation of the original, given equation. Except it includes the symbol for INTERSECTION not UNION. Be careful! All U's are not the same!

D. $P(A \cup B) > P(A) + P(B)$

Again the union is the set of all the items in A plus all the items in B, minus any overlap (if applicable). If there were *no* overlap, the two sides of this expression would be equal. Since that is possible, I know this is not something that MUST be true. In fact, it can't be true. If there *is* overlap, then $P(A) + P(B)$ will overestimate the value of the union (as it is the total before subtraction of the overlapping elements in the sets). Reversing the inequality sign and making it "or equal to" would make this expression true (i.e. $P(A \cup B) \leq P(A) + P(B)$).

E. Looking at the left of the equation, we see that we've simply multiplied the union by $P(A)$. Let's use substitution and expand this expression, using the information in the "Note":

$$P(A)P(A \cup B)$$

We now plug in $P(A) + P(B) - P(A)P(B)$ for $P(A \cup B)$ (this is given in the problem).

$$P(A)\left(P(A) + P(B) - P(A)P(B)\right)$$

Using the distributive property we get:

$$P(A)^2 + P(A)P(B) - P(A)^2 P(B)$$

Now we rearrange using the commutative property:

$$P(A)^2 - P(A)^2 P(B) + P(A)P(B)$$

Answer: **E.**

1. A box contains 10 blue marbles, 30 green marbles, and 24 orange marbles. How many orange marbles must be added so that there is a 60% chance of picking an orange marble at random?

A. 36
B. 40
C. 44
D. 48
E. 52

2. A number is chosen from the set $\{1,2,3,4,5,\ldots,24\}$. What is the probability that the number is a factor of 18?

A. $\frac{1}{2}$
B. $\frac{1}{3}$
C. $\frac{1}{4}$
D. $\frac{1}{5}$
E. $\frac{1}{6}$

3. There are 8 children, and you are assigned to line them up from youngest to oldest. You know which child is the oldest, which child is the second youngest, and which child is the youngest, but the other 5 children's ages are unknown. If you randomly sort the middle 5 children, what is the probability that all the children are ordered correctly?

A. $\frac{1}{20}$
B. $\frac{1}{120}$
C. $\frac{1}{760}$
D. $\frac{1}{5}$
E. $\frac{1}{25}$

4. In a set of integers from 1 to 50, inclusive, what is the probability of randomly selecting a prime number?

A. $\frac{1}{10}$
B. $\frac{1}{5}$
C. $\frac{2}{5}$
D. $\frac{3}{10}$
E. $\frac{1}{4}$

5. An integer from 10 through 999, inclusive, is to be chosen at random. What is the probability that the integer has 3 as 2 (not more) of its digits?

A. $\frac{1}{3}$
B. $\frac{1}{33}$
C. $\frac{11}{330}$
D. $\frac{2}{55}$
E. $\frac{3}{110}$

6. There are 200 paper slips in a hat, each numbered from $\sqrt{1},\sqrt{2},\ldots,\sqrt{200}$ with no repeats. What is the probability that the number on a slip drawn at random is not irrational?

A. $\frac{7}{100}$
B. $\frac{6}{100}$
C. $\frac{5}{100}$
D. $\frac{93}{100}$
E. $\frac{94}{100}$

7. O'Shea puts 7 blue marbles in a box. He now wants to add enough yellow marbles so that the probability of drawing a blue marble is $\frac{1}{11}$. How many yellow marbles does he need to add?

A. 50
B. 60
C. 70
D. 80
E. 90

8. At a party with 100 guests, there is a raffle. Each guest is given a ticket with a number from 00 to 99. There are no repeated numbers. Each guest signs his/her name on the ticket and drops it in a basket. There is a second basket of tickets numbered identically. A guest wins the raffle if his/her ticket is picked from the first basket and a ticket with the same ones digit is picked from the second basket. For example, if a guest has a ticket number 14 and the ticket picked from the first basket is 14 and the ticket from the second basket is 94, the guest wins. If Bernie's ticket number 42 is drawn from the first basket, what is the probability that Bernie will *not* win the raffle?

A. $\frac{1}{10}$
B. $\frac{9}{10}$
C. $\frac{1}{2}$
D. $\frac{3}{4}$
E. $\frac{2}{3}$

9. Robin has a fake coin that has a 40% chance of landing on tails and a 60% chance of landing on heads. In 4 coin tosses, what is the probability of getting exactly 3 tails?

A. .064
B. .936
C. .216
D. .784
E. .153

10. At a start-up company with a staff of 15 people, 6 people are male and 9 people are female. Two people are randomly chosen to be campus representatives. What is the probability that both representatives are male?

A. $\frac{6}{15}+\frac{6}{15}$
B. $\frac{6}{15}\cdot\frac{6}{15}$
C. $\frac{6}{15}\cdot\frac{5}{14}$
D. $\frac{9}{15}\cdot\frac{8}{14}$
E. $\frac{6}{15}\cdot\frac{5}{15}$

11. In a survey conducted at a university, students were asked to write down the number of campus organizations they are involved in. The results are shown below. What are the odds of a student at the university being involved in at least 3 organizations?

Distribution of Student Involvement in Campus Organizations					
# of organizations	0	1	2	3	>3
% of students	14	27	39	16	4

A. $1:25$
B. $1:5$
C. $1:4$
D. $1:20$
E. $4:25$

12. 25% of the dogs at a park are Corgis. There are 28 dogs at the park. How many dogs at the park are not Corgis?

A. 7
B. 14
C. 20
D. 21
E. 24

13. In a Secret Santa gift exchange, there are 3 gift cards, 5 stuffed animals, and 2 articles of clothing. 5 more people want to join the exchange. How many of the 5 people should bring stuffed animals so that the overall probability of getting a stuffed animal gift is 40%?

A. 1
B. 2
C. 3
D. 4
E. 5

14. Sam rolls a 6 sided die painted with 3 sides yellow, 2 sides red, and 1 side white. If Sam rolls the die and records the color of the side facing up repeatedly, how many times should Sam expect to record the color red after 180 rolls?

A. 30
B. 60
C. 90
D. 120
E. 150

15. A teacher lines 30 students in a single file line and starts passing out candy at the front of the line. The teacher has 15 lollipops, 10 candy canes, and 5 gumdrops. Lisa is 6^{th} in line to get the candy and the students in front of her have received 3 lollipops and 2 candy canes. What is the probability that Lisa will get a lollipop or a gumdrop?

A. $\frac{12}{25}$
B. $\frac{17}{30}$
C. $\frac{2}{3}$
D. $\frac{17}{25}$
E. $\frac{4}{5}$

16. In a list of 60 songs, there are 13 songs by artist A, 24 songs by artist B, 13 songs by artist C, and 10 songs by artist D. The first song on the playlist is set to play on random. What is the probability that the first song played is by artist B?

A. $\frac{1}{5}$
B. $\frac{13}{60}$
C. $\frac{2}{5}$
D. $\frac{3}{5}$
E. $\frac{4}{5}$

17. In box of 15 pebbles, 4 are white, 6 are black, and 5 are gray. If a blindfolded person is asked to pick one pebble out by random, what is the probability of the person picking a pebble that is not white?

A. $\frac{11}{15}$
B. $\frac{9}{15}$
C. $\frac{6}{15}$
D. $\frac{5}{15}$
E. $\frac{4}{15}$

18. Two events are independent if the outcome of one event does not affect the outcome of the other event. One of the following statements does NOT describe independent events. Which one?

A. An 8 is drawn from a deck of cards, then after replacing the card, an 8 is drawn.
B. An ace card is pulled from a deck of cards, then, without replacing the card a coin lands tails up.
C. A coin is flipped and lands heads up, then the same coin is flipped again and lands heads up.
D. A 4 is drawn from a deck of cards, then after replacing the card, a 3 is drawn.
E. A 4 is drawn from a deck of cards, then, without replacing the card, a king is drawn.

19. A new version of roulette is played where 2 pockets are green, 9 are red, 9 are black, 9 are blue, and 9 are yellow. If a ball is rolled into one of the pockets at random, what is the probability that it does NOT land in a blue pocket?

A. $\frac{29}{38}$

B. $\frac{27}{38}$

C. $\frac{9}{28}$

D. $\frac{3}{4}$

E. $\frac{1}{4}$

20. A taxi service has 240 taxis in its service. Based on previous data, the company constructed the table below showing the percent of taxis in use and the probabilities of occurring. Based on the probability distribution in the table, to the nearest whole number, what is the expected number of taxis that will be in use any given day?

Taxi Rate Usage	Probability
0.4	0.3
0.6	0.4
0.7	0.2
0.9	0.1

A. 59
B. 60
C. 142
D. 144
E. 156

21. For the first 7 possible values of x, the table below gives the probability, $P(x)$, that x inches of rain, to the nearest inch will fall in any given month.

x inches of rain	$P(x)$
0	0.3102
1	0.1020
2	0.1567
3	0.2021
4	0.1166
5	0.0621
6	0.0503

Which of the following values is closest to the probability that at least 3 inches of rain will fall in any given month?

A. 0.11
B. 0.20
C. 0.40
D. 0.43
E. 0.57

22. Let X and Y be independent events. $P(x)$ represents the probability that event X will occur, $P(\sim x)$ represents the probability that event x will not occur, and $P(x \cap y)$ represents the probability that both events X and Y will occur. Which of the following equations *must* be true?

A. $P(x) = P(y)$

B. $P(x \cap \sim y) = P(\sim x \cap y)$

C. $P(x) - P(\sim x) = P(y) - P(\sim y)$

D. $P(x \cap y) = P(\sim x \cap \sim y)$

E. $P(x) \geq P(x \cap \sim y)$

23. A lab is testing a new machine to diagnose breast cancer. In 50 trials of 800 individuals, the number of false positives (instances when the machine diagnoses a woman with breast cancer who does not actually have it) were recorded. Based on the distribution below, what is the expected number of false positives that will occur among $50,000$ tests?

Number, n, of false positives	Probability that n false positives are produced in a trial of 800 people
0	0.2
1	0.4
2	0.15
3	0.15
4	0.1

A. 1.55
B. 63
C. 97
D. 124
E. 500

24. The probability distribution of the discrete random variable Y is shown in the table below. What is closest to the expected value of Y?

y	$P(Y=y)$
0	$\frac{2}{9}$
1	$\frac{1}{18}$
2	$\frac{5}{18}$
3	$\frac{1}{6}$
4	$\frac{2}{9}$
5	$\frac{1}{18}$

A. 1
B. 2
C. 2.28
D. 3
E. $\frac{2}{9}$

25. The table below shows the results of a survey of 300 people who were asked whether they liked spicy food and whether they liked hiking.

	Like spicy food	Do not like spicy food	Total
Like to hike	75	115	185
Do not like to hike	40	75	115
Total	110	190	300

According to the results, which is closest to the probability that a randomly selected person who was surveyed doesn't like spicy food given that they don't like to hike?

A. 165%
B. 65%
C. 63%
D. 39%
E. 38%

26. The probability that a specific event, E, happens is denoted $P(E)$. The probability that this event does not happen is denoted $P(not\ E)$. Which of the following statements is *always* true?

A. $P(not\ E) > P(E)$
B. $P(not\ E) < P(E)$
C. $P(not\ E) = P(E) + 1$
D. $1 - P(E) = P(not\ E)$
E. $0 < P(not\ E) < P(E)$

27. Suppose that a will be randomly selected from the set $\{-3,-1,0,1,2\}$ and that b will be randomly selected from the set $\{-3,-2,0,1,2,3\}$. What is the probability that $ab < 0$?

A. $\frac{1}{3}$
B. $\frac{3}{20}$
C. $\frac{13}{30}$
D. $\frac{4}{15}$
E. $\frac{1}{15}$

28. Best friends Mylah and Sierra and three other classmates have been instructed to stand in a straight line in a randomly assigned order.What is the probability that Mylah and Sierra will stand next to each other?

A. $\frac{2}{5}$

B. $\frac{1}{5}$

C. $\frac{1}{15}$

D. $\frac{1}{30}$

E. $\frac{1}{120}$

ANSWER KEY

1. A	**2. C**	**3. B**	**4. D**	**5. E**	**6. A**	**7. C**	**8. B**	**9. E**	**10. C**	**11. C**	**12. D**	**13. A**	**14. B**
15. D	**16. C**	**17. A**	**18. E**	**19. A**	**20. C**	**21. D**	**22. E**	**23. C**	**24. C**	**25. B**	**26. D**	**27. A**	**28. A**

ANSWER EXPLANATIONS

1. **A.** We have a total of $10+30+24=64$ marbles, 24 of them are orange marbles, and we are looking to find the number of additional orange marbles to add in order to have a 60% probability of picking an orange marble. Let x be the number of additional orange marbles needed. Then, we can say that the probability of picking an orange marble after the addition of the x marbles is $\frac{24+x}{64+x}$, because we'll be adding orange marbles to the orange marbles (numerator) and to the total number of marbles (denominator). We want this to be equal 60%, which equals 0.60 or 6/10, so we set up the equation $\frac{24+x}{64+x}=\frac{6}{10}$. Cross-multiplying and distributing gives us $240+10x=384+6x$. Then, subtracting $6x$ from both sides, we get $240+4x=384$. Subtracting 240 from both sides: $4x=144$. Finally, dividing each side by 4 we find $x=36$. So, we need to add 36 additional orange marbles for there to be a 60% chance of picking an orange marble.

2. **C.** First, we must find all factors of 18 (integers that 18 can be divided by). (See LCM/GCF Chapter for help with factoring using a factor rainbow). They are $1,2,3,6,9$, and 18. We see that 18 has 6 factors and all of these numbers are included in the set $\{1,2,3,4,....,24\}$. The set $\{1,2,3,4,....,24\}$ has 24 numbers, so the probabilty of choosing a factor of 18 from these 24 numbers is $\frac{6}{24}=\frac{1}{4}$.

3. **B.** In order to line the children up from youngest to oldest, we start with the youngest. Since we already know which children are the two youngest and their ages, we know which one child to place in the first spot, and which one child to place in the second spot. The same goes for the last spot since we know which child is the oldest. For the remaining spots $3-7$, we have 5 children left who need to be placed. For spot three there are 5 children who could randomly be placed there. Once one child is randomly placed in the third spot, there are 4 children left who could be placed in the fourth spot, and then 3 children for the fifth spot, 2 choices for the sixth spot, and one remaining child at the end who will take the 7^{th} spot. So, the total possible ways of ordering the middle five children is calculated $1\times1\times5\times4\times3\times2\times1\times1=120$. Since only one of these line-ups is the correct order, the probability that the children are ordered correctly is $\frac{1}{120}$.

4. **D.** We must first list all the prime numbers between 1 and 50. $2,3,5,7,11,13,17,19,23,29,31,37,41,43,47$. 15 out of the 50 integers from 1 to 50, inclusive, are primes. (Remember, 1 is not prime!) So, the probability of selecting a prime number is $\frac{15}{50}=\frac{3}{10}$. One way to determine if a number is prime is to divide that number by all integers less than or equal to its square root, and if the number cannot be divided by any of these integers other than 1, it is prime. For example, 13 is prime because the square root of 13 is approximately 3.60555, and 13 is not divisible by 3 or 2. See Ch 2 for more on prime numbers.

5. **E.** First, we determine that there are $999-10+1=990$ integers from 10 through 999 (we add 1 because the set is inclusive). (Alternatively, we can reason that we take away numbers $1-9$ from the 999 numbers included in $1-999$ and that leaves 990). Then we count the number of integers that have 3 as exactly two of their digits. If we let x represent any digit from $0-9$ <u>excluding</u> 3 then the numbers we want to count can be represented in the following forms: $x33$, $3x3$, and $33x$. Since there are 9 possible choices for x $(0, 1, 2, 4, 5, 6, 7, 8, 9)$ while the other two digits in our numbers only have one possible choice respectively (3), the number of possible permutations could be calculated for each of the forms. $x33$ has $1\times9\times1=9$ possible outcomes, $3x3$ has $1\times9\times1=9$ possible outcomes, and $33x$ has $1\times1\times9=9$ possible outcomes. This gives us a total of $9+9+9=27$ numbers that satisfy our condition of having 3 as

2 of its digits. So, the probability of choosing such an integer out of a total of 990 numbers (calculated earlier) is $\frac{27}{990} = \frac{3}{110}$.

6. **A.** To be "not irrational" is the same as to be rational. Since there are 200 integers from 1 to 200, there are also 200 values from $\sqrt{1}, \sqrt{2}, \ldots, \sqrt{200}$. We must now find the number of values from $\sqrt{1}, \sqrt{2}, \ldots, \sqrt{200}$ that are rational. An rational number is a number that can be expressed as a fraction or in the form $\frac{n}{d}$ where n and d are integers. Numbers of the form $\sqrt{x}$ are rational only if x is a perfect square. So, since there are 14 numbers from 1 to 200 that are perfect squares $(1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, \text{ and } 196)$, the probability of drawing one is $\frac{14}{200}$ or $\frac{7}{100}$.

7. **C.** The box initially only has 7 blue marbles. Let x be the number of yellow marbles we want to add in order for the probability of drawing a blue marble to be $\frac{1}{11}$. This means $\frac{7}{7+x} = \frac{1}{11}$. Cross-multiplying this equation, we get $77 = 7 + x$. Subtracting 7 from both sides, we get $70 = x$. Accordingly, we need to add 70 yellow marbles in order for the probability of drawing a blue marble to be equal to $\frac{1}{11}$.

8. **B.** Bernie will not win the raffle if the ticket drawn from the second basket does not end in the same ones digit as 42. So, if the second ticket does not end with a 2, Bernie will not win. The possible ticket numbers that will not allow Bernie to win have 10 possible numbers $(0-9)$ for the first digit and 9 possible numbers ($0-9$ excluding 2) for the second digit (ones place). This gives $10 \times 9 = 90$ possible outcomes out of a total of $99 - 0 + 1 = 100$ tickets to choose from for Bernie to lose. The probability of choosing such a ticket is $\frac{90}{100} = \frac{9}{10}$.

9. **E.** There are 4 orders for which Robin can get 3 tails out of 4 tosses: TTTH, TTHT, THTT, or HTTT. First we find the probability of each possible outcome. Each coin toss is an independent "AND" event, so the probability that she will get, for example, tails in the first toss *AND* tails in the second toss *AND* heads in the third toss *AND* tails in the fourth toss, is found by multiplying the probability of each independent coin toss. The probability of each of these outcomes is, respectively, $(0.4)(0.4)(0.4)(0.6)$, $(0.4)(0.4)(0.6)(0.4)$, $(0.4)(0.6)(0.4)(0.4)$, and $(0.6)(0.4)(0.4)(0.4)$. Now, the probability that she will get one of the four desired outcomes is an independent "OR" event, the probability that she will get TTTH *OR* TTHT *OR* THTT *OR* HTTT, so we must sum the individual probabilities we found before. Thus, the total probability of landing tails 3 out of 4 times is $(0.4)(0.4)(0.4)(0.6) + (0.4)(0.4)(0.6)(0.4) + (0.4)(0.6)(0.4)(0.4) + (0.6)(0.4)(0.4)(0.4)$, or $4 \cdot (0.4)^3 (0.6)$. This simplifies to $4 \times 0.0384 = 0.154$.

10. **C.** When calculating the probability of choosing two males, we are looking for the probability of selecting one male <u>and</u> selecting another male from the <u>remaining</u> staff. Thus these are dependent events. The probability of selecting the first male is $\frac{6}{15}$. Taking out the first selected male, the pool of candidates now consists of 5 males and 9 females, so the probability of selecting a second male from the staff not including the first male is $\frac{5}{14}$. We multiply the two probabilities to find the probability of both occurrences happening, so the probability that both representatives are male is $\frac{6}{15} \times \frac{5}{14}$. A common mistake made is multiplying $\frac{6}{15} \times \frac{6}{15}$. This is the probability of selecting two males with replacement, which means that it is possible to select the same person twice. Since the staff is selecting two different people, they are selecting without replacement.

11. **C.** Most students miss this question because they don't know what "odds" means. Odds are expressed as part to part (ratio) not as a fraction (part of the whole.) The odds of a student being involved in at least 3 organizations is the percentage of the student being in 3 or more clubs against the percentage of the student being in 0, 1, or 2. The percentage of a student being in 3 *or* more than 3 clubs is $16\% + 4\% = 20\%$ and the percentage of students in the remaining tallies are $14\% + 27\% + 39\% = 80\%$. So, the odds are $20:80$, which reduces to $1:4$. Note the word "odds" is rare on the ACT.

12. D. Out of the 28 dogs, 25% or $\frac{28}{4}=7$ are Corgis. The number of dogs at the park that are NOT Corgis is $28-7=21.$

13. A. Initially, there are a total of $3+5+2=10$ gifts, and 5 of these gifts are stuffed animals. With the addition of 5 more gifts, the total number of gifts in the exchange is now $10+5=15.$ If we want 40% of those 15 gifts to be stuffed animals, then $0.4\times15=6$ of the gifts must be stuffed animals. We already know that there are 5 stuffed animal gifts from the original 10 people, so we only need $6-5=1$ out of the 5 people joining to bring a stuffed animal gift.

14. B. To find the expected number of rolls that are red side up, we must first determine the probability of rolling a red. Since 2 out of the 6 sides are red, the probability of rolling a red is $\frac{2}{6}=\frac{1}{3}$. The expected number of rolls that are red side up is then the probability of rolling a red multiplied by the total number of rolls made. Out of the total 180 rolls, $\frac{1}{3}$ of them are expected to be red: $180\times\frac{1}{3}=60$ rolls.

15. D. This is an "OR" situation so we calculate the independent probabilities and add them together. The teacher started out with 15 lollipops, 10 candy canes, and 5 gumdrops. At the time the teacher reaches Lisa, there are $15-3=12$ lollipops left, $10-2=8$ candy canes left, and $5-0=5$ gumdrops remaining. The total number of candies by the time the teacher reaches Lisa is now $30-5=25$. So the probabilities of getting a lollipop, candy cane, and gumdrop are $\frac{12}{25}$, $\frac{8}{25}$, and $\frac{5}{25}$ respectively. The probability of getting a lollipop **<u>or</u>** a gumdrop is $\frac{12}{25}+\frac{5}{25}=\frac{17}{25}$.

16. C. There are 24 songs out of 60 that are by artist B, so the probability of the first song to be by artist B is $\frac{24}{60}=\frac{12}{30}=\frac{6}{15}=\frac{2}{5}$.

17. A. We wish to find the probability of the pebble being NOT white, which is $1-$ (probability of picking a white pebble). The probability of picking a white pebble is $\frac{4}{15}$, so the probability of picking a picking a pebble that is *not* white is $1-\frac{4}{15}=\frac{15}{15}-\frac{4}{15}=\frac{11}{15}$.

18. E. For each answer choice, the events described are independent except for answer choice E because if a 4 is drawn from a deck, and without replacement, a king is drawn, then the probability of drawing the king depends on whether or not the first card drawn from the deck was also a king. All other answer choices describe events that are independent because each draw made from a deck is replaced, making the next draw not dependent on the previous draw. Each coin toss also does not depend on the previous toss.

19. A. The probability that an event does not occur is equal to the sum of the probabilities of all other alternative possibilities added together. However, since we may come across a problem where there are too many alternate events to calculate in the time we have, it's better to calculate the probability that an event will not happen as $100\%,$ or $1,$ minus the probability that the event will happen. In this case, that is $1-\frac{9}{38}$, since there are 9 chances for the ball to land in blue out of 38 possibilities. This gives us $\frac{38}{38}-\frac{9}{38}=\frac{29}{38}$.

20. C. The expected value is equal to the sum of all possible values, each multiplied by its probability. Our expected taxi usage rate is $0.4(0.3)+0.6(0.4)+0.7(0.2)+0.9(0.1)=0.59$. We expect the taxi service to be using 59% of its taxis at any given time. To find the expected number of taxis, not the rate of taxi use, we multiply the number of taxis, $240,$ by the rate of use: $240(59\%)=240(0.59)=141.6\approx142$.

21. D. Be careful: we need the probability that *at least* 3 inches of rain will fall. But the chart ONLY lists the **first SEVEN possibilities**. Thus we don't know the probability for more than 6 inches of rain. Our best bet is NOT to try to find the sum of **all** probabilities of $x\geq3$ in the chart, because the chart leaves off some values. If you tried this method, you would get choice C. Instead, find the probability that this WON'T happen and subtract from one. Doing so will account for the missing chart values. First, take the sum of the probabilities for less than 3 inches of rain. $0.3102+0.1020+0.1567=0.5689$ Now, subtract that value from one: $1-0.5689=0.4311$ The closest value is answer choice (D). Choice B is incorrect as it only is for three inches of rain, not all values equal to or greater than three inches.

22. **E.** The likelihood of a given event happening is always greater than or equal to the probability of that event happening alongside a second event (and only equal when the probability of the second event is 100%!) Remember, an upside down U shape means intersection, or both events have occured. This is the only choice that <u>MUST</u> be true.

23. **C.** The expected number of false positives in a group of 800 people will be equal to the sum of each number n of false positives times the probability of each potential outcome in the sample. The expected number in the <u>sample</u> is $0(0.2)+1(0.4)+2(0.15)+3(0.15)+4(0.1)=\ 0+0.4+0.3+0.45+0.4=1.55.$ However, we are looking for the number of false positives in a <u>population</u> of $50{,}000$ people, so we can use a proportion to project how many people this would be given the rate of false positives in the sample. We set false positives in the sample over the total in the sample equal to false positive in the population (n) over the total population: $\frac{1.55}{800}=\frac{n}{50{,}000}$ Cross multiplying, we find $1.55(50{,}000)=800n$ or $\frac{1.55(50{,}000)}{800}=n=96.875$ which is ≈ 97 false positives.

24. **C.** The expected value of a variable is the sum of all of its possible values multiplied by their respective probabilities. The expected value of Y equals $0\left(\frac{2}{9}\right)+1\left(\frac{1}{18}\right)+2\left(\frac{5}{18}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{2}{9}\right)+5\left(\frac{1}{18}\right)=\frac{0}{9}+\frac{1}{18}+\frac{10}{18}+\frac{3}{6}+\frac{8}{9}+\frac{5}{18}\approx 2.28$. Just because our inputs are discrete random variables (i.e. integers) doesn't mean round our expected value to an integer. The expected value is a weighted mean or average so should not be rounded to the nearest integer unless specified.

25. **B.** Since we are given that the person doesn't like to hike, we can get rid of all of the individuals who do like to hike, leaving us with a population of 115. This is our denominator. In this group, 75 people don't like spicy food, so our probability is $\frac{75}{115}\approx 0.65=65\%$.

26. **D.** The probability of an event occurring and the probability of it not occurring must always equal $1,$ since those are the only two possible outcomes. Thus we know that $1=P(E)+P(\mathit{not}\ E)$, which makes $1-P(E)=P(\mathit{not}\ E)$ also true.

27. **A.** The probability that $ab<0$ is equal to the probability that $a<0$ and $b>0$ plus the probability that $a>0$ and $b<0$. There are 5 possibilities for a, of which 2 are negative and 2 are positive. There are 6 possibilities for b, of which 2 are negative and 3 are positive. The probability that $a<0$ and $b>0=\frac{2}{5}\times\frac{3}{6}=\frac{6}{30}=\frac{1}{5}.$ The probability that $a>0$ and $b<0=\frac{2}{5}\times\frac{2}{6}=\frac{4}{30}=\frac{2}{15}.$ Now we add these two probabilities together: $\frac{1}{5}+\frac{2}{15}=\frac{3}{15}+\frac{2}{15}=\frac{5}{15}=\frac{1}{3}$

28. **A.** Here we'll use the principles of arrangements. Remember probability is always:

$$\frac{\text{the number of desired outcomes}}{\text{the number of possible outcomes}}$$

In this line, order matters. Three other classmates means FIVE total. We'll have to manually figure out the numerator but we can use permutations to figure out the denominator. First calculate the number of possible straight line arrangements: 5 people taken 5 at a time (${}_5P_5$) or: $5!=5\times4\times3\times2\times1=120.$ This is our denominator. Now we need to count the number of desired outcomes for the numerator. Below, 0 denotes other kids and M and S denote Mylah and Sierra:

If Mylah is first and Sierra after her, we have four options: MS000 0MS00 00MS0 000MS

If Sierra is first and Mylah is second, we have four more: SM000 0SM00 00SM0 000SM

We don't care what position the other kids are in—but we do have to account for them. If there are three other positions to fill, we have $3\times2\times1$ options for arranging those three spots—or in other words each of the above "codes" actually stands for SIX different possibilities. So we need to take the 8 arrangements and multiply each of them by 6—because regardless of where the three open slots are—these three slots represent 6 different orientations. That gives us:

$$\frac{8\times6}{5\times4\times3\times2\times1}$$

We can cancel the 6, and then reduce by dividing out 4:

$$\frac{8\times\cancel{6}}{5\times4\times\cancel{3}\times\cancel{2}\times1}=\frac{8}{20}=\frac{2}{5}$$

This problem is significantly more difficult than the majority of probability problems you'll find on the ACT®. If you can do this, you are set!